11.9 The field of a permanent magnet

The uniformly polarized spheres and rods we talked about in Chapter 10 are seldom seen, even in the laboratory. Frozen-in electric polarization can occur in some substances, although it is usually disguised by some accumulation of free charge. To make Fig. 11.3(a), which shows how the field of a polarized rod *would* look, it was necessary to use two charged disks. On the other hand, materials with permanent magnetic polarization, that is, permanent *magnetization,* are familiar and useful. Permanent magnets can be made from many alloys and compounds of ferromagnetic substances. What makes this possible is a question we leave for Section 11.11, where we dip briefly into the physics of ferromagnetism. In this section, taking the existence of permanent magnets for granted, we want to study the magnetic field **B** of a uniformly magnetized cylindrical rod and compare it carefully with the electric field **E** of a uniformly polarized rod of the same shape.

Figure 11.22 shows each of these solid cylinders in cross section. The polarization, in each case, is parallel to the axis, and it is uniform. That is, the polarization **P** and the magnetization **M** have uniform magnitude and direction everywhere within their respective cylinders. In the magnetic case this implies that every cubic millimeter of the permanent magnet has the same number of lined-up electron spins, pointing in the same direction. (A very good approximation to this can be achieved with modern permanent magnet materials.)

By the field inside the cylinder we mean, of course, the macroscopic field defined as the space average of the microscopic field. With this understanding, we show in Fig. 11.22 the field lines both inside and

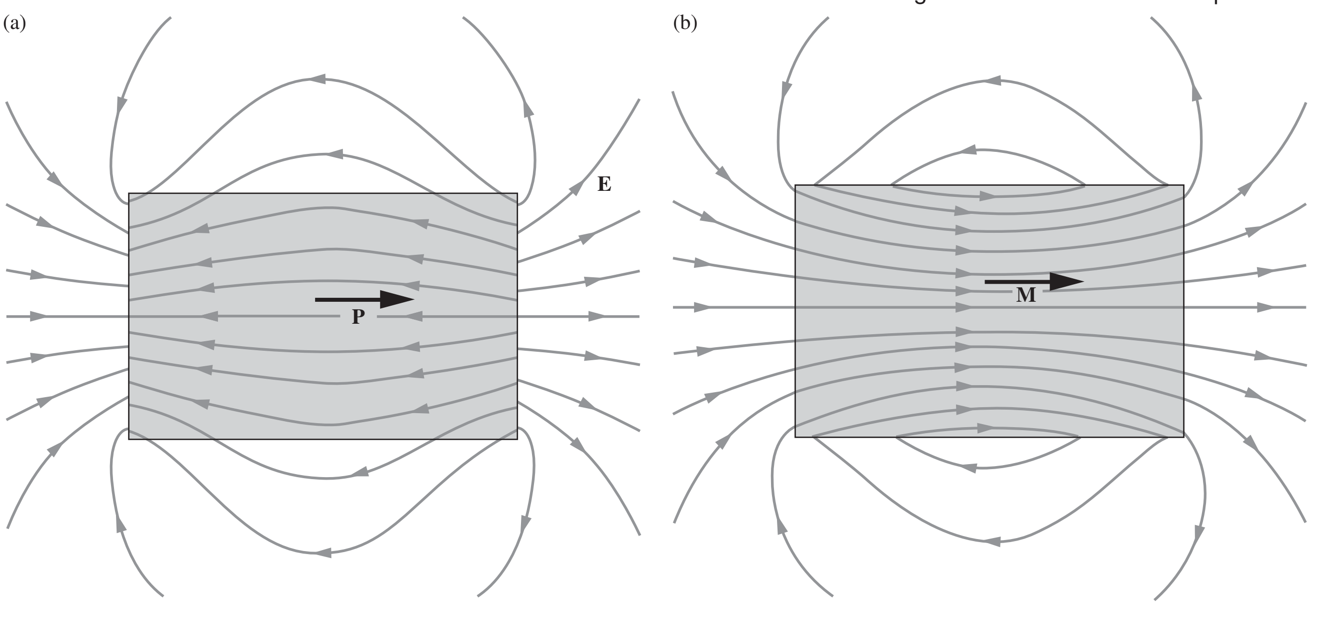


Figure 11.22.
(a) The electric field **E** outside and inside a uniformly polarized cylinder. (b) The magnetic field **B** outside and inside a uniformly magnetized cylinder. In each case, the interior field shown is the macroscopic field, that is, the local average of the atomic or microscopic field.

outside the rods. By the way, these rods are not supposed to be near one another; we only put the diagrams together for convenient comparison. Each rod is isolated in otherwise field-free space. (Which do you think would more seriously disturb the field of the other, if they *were* close together?)

Outside the rods the fields **E** and **B** *look alike*. In fact the field lines follow precisely the same course. That should not surprise you if you recall that the electric dipole and the magnetic dipole have similar far fields. Each little chunk of the magnet is a magnetic dipole, each little chunk of the polarized rod (sometimes called an *electret*) is an electric dipole, and the field outside is the superposition of all their far fields.

The field **B**, inside and out, is the same as that of a cylindrical sheath of current. In fact, if we were to wind very evenly, on a cardboard cylinder, a single-layer solenoid of fine wire, we could hook a battery up to it and duplicate the exterior and interior field **B** of the permanent magnet. (The coil would get hot and the battery would run down, whereas electron spins provide the current free and frictionless!) The electric field **E**, both inside and outside the polarized rod, is that of two disks of charge, one at each end of the cylinder.

Observe that the *interior* fields **E** and **B** are essentially different in form: **B** points to the right, is continuous at the ends of the cylinder, and suffers a sharp change in direction at the cylindrical surface. (These three facts are consistent with the **B** field that arises from a cylindrical sheath of current.) On the other hand, **E** points to the left, passes through the cylindrical surface as if it weren't there, but is discontinuous at the end surfaces. (These three facts are consistent with the **E** field that arises from two disks of charge.) These differences arise from the essential difference between the "inside" of the physical electric dipole and the "inside" of the physical magnetic dipole seen in Fig. 11.8. By *physical,* we mean the ones Nature has actually provided us with.

If the external field were our only concern, we could use either picture to describe the field of our magnet. We could say that the magnetic field of the permanent magnet arises from a layer of positive magnetic charge – a surface density of north magnetic poles on the right-hand end of the magnet, and a layer of negative magnetic charge, south poles, on the other end. We could adopt a scalar potential function ϕ_{mag}, such that $\mathbf{B} = -\text{grad}\,\phi_{\text{mag}}$. The potential function ϕ_{mag} would be related to the fictitious pole density as the electric potential is related to charge density. The simplicity of the scalar potential compared with the vector potential is rather appealing. Moreover, the magnetic scalar potential can be related in a very neat way to the currents that are the real source of **B**, and thus one can use the scalar potential without any explicit use of the fictitious poles. You may want to use this device if you ever have to design magnets or calculate magnetic fields.

We must abandon the magnetic pole fiction, however, if we want to understand the field inside the magnetic material. That the macroscopic magnetic field inside a permanent magnet is, in a very real sense, like the field in Fig. 11.22(b) rather than the field in Fig. 11.22(a) has been demonstrated experimentally by deflecting energetic charged particles in magnetized iron, as well as by the magnetic effects on slow neutrons, which pass even more easily through the interior of matter.

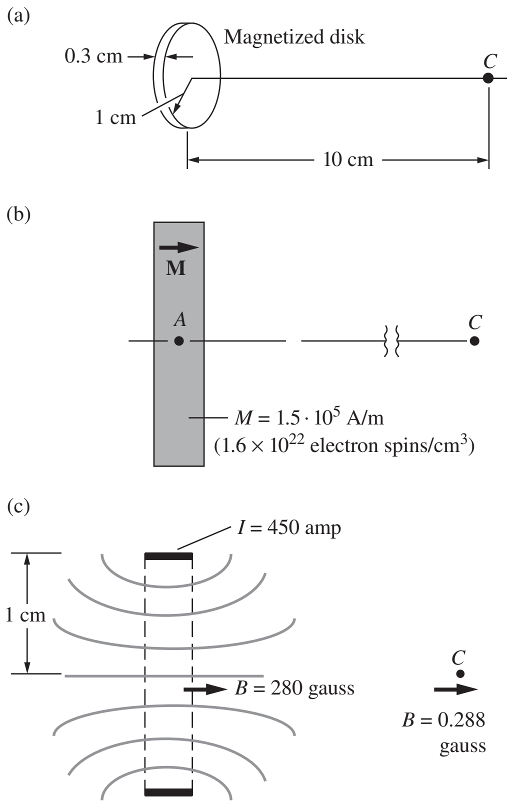


Figure 11.23.
(a) A disk uniformly magnetized parallel to its axis. (b) Cross-sectional view of disk. (c) The equivalent current is a band of current amounting to 450 amps flowing around the rim of the disk. The magnetic field B is the same as the magnetic field of a very short solenoid, or approximately that of a simple ring of current of 1 cm radius.

Example (Disk magnet) Figure 11.23(a) shows a small disk-shaped permanent magnet, in which the magnetization is parallel to the axis of symmetry. Although many permanent magnets take the form of bars or horseshoes made of iron, flat disk magnets of considerable strength can be made with certain rare-earth elements. The magnetization M is given as $1.5 \cdot 10^5$ joules/(tesla-m^3), or equivalently amps/meter. The magnetic moment of the electron is $9.3 \cdot 10^{-24}$ joule/tesla, so this value of M corresponds to $1.6 \cdot 10^{28}$ lined-up electron spins per cubic meter. The disk is equivalent to a band of current around its rim, of surface density $\mathcal{J} = M$. The rim being $\ell = 0.3$ cm wide, the current I amounts to

$$I = \mathcal{J}\ell = M\ell = (1.5 \cdot 10^5 \text{ amp/m})(3 \cdot 10^{-3} \text{ m}) = 450 \text{ amps}. \tag{11.62}$$

This is rather more current than you draw by short-circuiting an automobile battery! The field **B** at any point in space, including points inside the disk, is simply the field of this band of current. For instance, near the center of the disk, **B** is approximately (using Eq. (6.54))

$$B = \frac{\mu_0 I}{2r} = \frac{(4\pi \cdot 10^{-7} \text{ kg m/C}^2)(450 \text{ C/s})}{2(0.01 \text{ m})} = 2.8 \cdot 10^{-2} \text{ tesla}, \tag{11.63}$$

or 280 gauss. The approximation consists in treating the 0.3 cm wide band of current as if it were concentrated in a single thin ring. (The corresponding approximation in an electrical setup would be to treat the equivalent charge sheets as large compared with their separation.) As for the field at a distant point, it would be easy to compute it for the ring current, but we could also, for an approximate calculation, proceed as we did in the electrical example. That is, we could find the total magnetic moment of the object, and find the distant field of a single dipole of that strength.

11.10 Free currents, and the field H

It is often useful to distinguish between bound currents and free currents. *Bound* currents are currents associated with molecular or atomic magnetic moments, including the intrinsic magnetic moment of particles with spin. These are the molecular current loops envisioned by Ampère, the source of the magnetization we have just been considering. *Free* currents are ordinary conduction currents flowing on macroscopic paths – currents that can be started and stopped with a switch and measured with an ammeter.

The current density $\mathbf{J}$ in Eq. (11.56) is the macroscopic average of the bound currents, so let us henceforth label it $\mathbf{J}_{\text{bound}}$:

$$\mathbf{J}_{\text{bound}} = \text{curl}\,\mathbf{M}. \tag{11.64}$$

At a surface where $\mathbf{M}$ is discontinuous, such as the side of the magnetized block in Fig. 11.17, we have a surface current density $\mathcal{J}$ which also represents bound current.

We found that $\mathbf{B}$, both outside matter and, as a space average, inside matter, is related to $\mathbf{J}_{\text{bound}}$ just as it is to any current density. That is, $\text{curl}\,\mathbf{B} = \mu_0 \mathbf{J}_{\text{bound}}$. But that was in the absence of free currents. If we bring these into the picture, the field they produce simply adds on to the field caused by the magnetized matter and we have

$$\text{curl}\,\mathbf{B} = \mu_0(\mathbf{J}_{\text{bound}} + \mathbf{J}_{\text{free}}) = \mu_0 \mathbf{J}_{\text{total}}. \tag{11.65}$$

Let us express $\mathbf{J}_{\text{bound}}$ in terms of $\mathbf{M}$, through Eq. (11.64). Then Eq. (11.65) becomes

$$\text{curl}\,\mathbf{B} = \mu_0(\text{curl}\,\mathbf{M}) + \mu_0 \mathbf{J}_{\text{free}}, \tag{11.66}$$

which can be rearranged as

$$\text{curl}\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_{\text{free}}. \tag{11.67}$$

If we now *define* a vector function $\mathbf{H}(x, y, z)$ at every point in space by the relation

$$\boxed{\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}} \tag{11.68}$$

then Eq. (11.67) can be written as

$$\boxed{\text{curl}\,\mathbf{H} = \mathbf{J}_{\text{free}}} \tag{11.69}$$

In other words, the vector $\mathbf{H}$, defined by Eq. (11.68), is related (up to a factor of μ_0) to the *free* current in the way $\mathbf{B}$ is related to the total current, *bound* plus *free*. The parallel is not complete, however, for we always have $\text{div}\,\mathbf{B} = 0$, whereas our vector function $\mathbf{H}$ does not necessarily have zero divergence.

This surely has reminded you of the vector $\mathbf{D}$ which we introduced, a bit grudgingly, in Chapter 10. Recall that $\mathbf{D}$ is related (up to a factor of ϵ_0) to the free charge as $\mathbf{E}$ is related to the total charge. Although we rather disparaged $\mathbf{D}$, the vector $\mathbf{H}$ is really useful, for a practical reason that is worth understanding. In short, the reason is that in our two equations, $\text{div}\,\mathbf{D} = \rho_{\text{free}}$ and $\text{curl}\,\mathbf{H} = \mathbf{J}_{\text{free}}$, the charge density ρ_{free} is difficult to measure, while the current density $\mathbf{J}_{\text{free}}$ is easy. Let's look at this in more detail.

In electrical systems, what we can easily control and measure are the potential differences of bodies, and not the amounts of free charge on them. Thus we control the electric field **E** directly. **D** is out of our direct control, and since it is not a fundamental quantity in any sense, what happens to it is not of much concern. In magnetic systems, however, it is precisely the free currents that we can most readily control. We lead them through wires, measure them with ammeters, channel them in well-defined paths with insulation, and so on. We have much less direct control, as a rule, over magnetization, and hence over **B**. So the auxiliary vector **H** *is* useful, even if **D** is not.

The integral relation equivalent to Eq. (11.69) is

$$\boxed{\int_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_{\text{free}}\cdot d\mathbf{a} = I_{\text{free}}} \tag{11.70}$$

where I_{free} is the total free current enclosed by the path C. Suppose we wind a coil around a piece of iron and send through this coil a certain current I, which we can measure by connecting an ammeter in series with the coil. This is the free current, and it is the only free current in the system. Therefore one thing we know for sure is the line integral of **H** around any closed path, whether that path goes through the iron or not. The integral depends only on the number of turns of our coil that are linked by the path, and not on the magnetization in the iron. The determination of **M** and **B** in this system may be rather complicated. It helps to have singled out one quantity that we can determine quite directly. Figure 11.24 illustrates this property of **H** by an example, and is a reminder of the units we may use in a practical case. **H** has the same units as $\mathbf{B}/\mu_0$, or equivalently the same units as **M**, which are amps/meter. This is consistent with the fact that curl **H** equals $\mathbf{J}_{\text{free}}$, which has units of amps/meter^2. It is also consistent with the fact that $\int \mathbf{H}\cdot d\mathbf{l}$ equals I_{free}, which has units of amps.

We consider **B** the fundamental magnetic field vector because the absence of magnetic charge, which we discussed in Section 11.2, implies $\text{div}\,\mathbf{B} = 0$ everywhere, even inside atoms and molecules. From $\text{div}\,\mathbf{B} = 0$ it follows, as we showed in Section 11.8, that the average macroscopic field inside matter is **B**, not **H**. The implications of this have not always been understood or heeded in the past. However, **H** has the practical advantage we have already explained. In some older books you will find **H** introduced as the primary magnetic field. **B** is then defined as $\mu_0(\mathbf{H} + \mathbf{M})$, and given the name *magnetic induction*. Even some writers who treat **B** as the primary field feel obliged to call it the magnetic induction because the name *magnetic field* was historically preempted by **H**. This seems clumsy and pedantic. If you go into the laboratory and ask a physicist what causes the pion trajectories in his bubble chamber to curve, you will probably receive the answer "magnetic field," not "magnetic induction." You will seldom hear a geophysicist refer to the earth's magnetic induction, or an astrophysicist talk about the magnetic induction in the galaxy. We propose to keep on calling **B** the magnetic field.

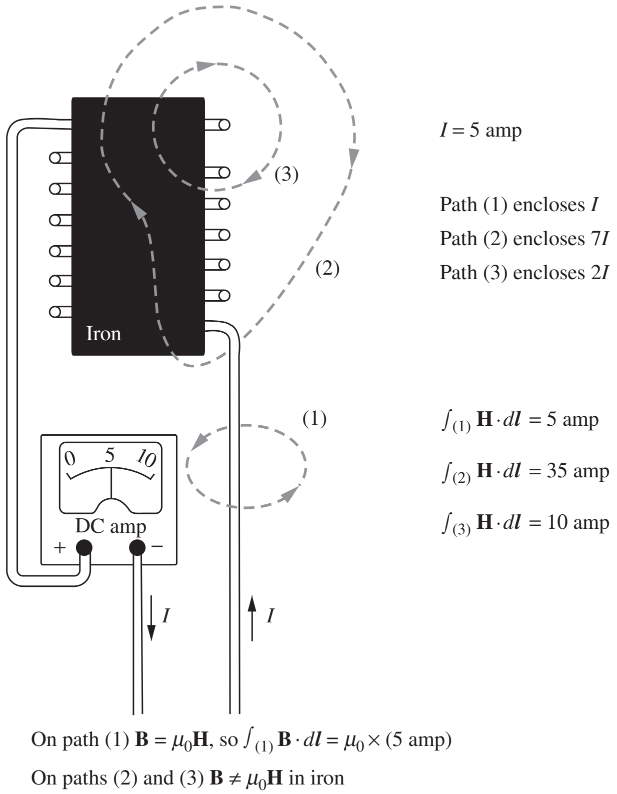


Figure 11.24.
Illustrating the relation between free current and the line integral of **H**.

As for **H**, although other names have been invented for it, we shall call it *the field* **H**, or even *the magnetic field* **H**.

It is only the names that give trouble, not the symbols. Everyone agrees that in the SI system the relation connecting **B**, **M**, and **H** is that stated in Eq. (11.68). In empty space we have $\mathbf{H} = \mathbf{B}/\mu_0$, for **M** must be zero where there is no matter.

In the description of an electromagnetic wave, it is common to use **H** and **E**, rather than **B** and **E**, for the magnetic and electric fields. For the plane wave in free space that we studied in Section 9.4, the relation between the magnetic amplitude H_0 in amps/meter and the electric amplitude E_0 in volts/meter involves the constant $\sqrt{\mu_0/\epsilon_0}$ which has the dimensions of resistance and the approximate value 377 ohms. For its exact value, see Appendix E. We met this constant before in Section 9.6, where it appeared in the expression for the power density in the plane wave, Eq. (9.36). The condition that corresponds to E_0 and B_0, as stated by Eq. (9.26), becomes

$$E_0(\text{volt/meter}) = H_0(\text{amp/meter}) \times 377 \text{ ohms}. \qquad (11.71)$$

This makes a convenient system of units for dealing with electromagnetic fields in vacuum whose sources are macroscopic alternating currents and voltages. But remember that the basic magnetic field *inside* matter is **B**, not **H**, as we found in Section 11.9. That is not a matter of mere definition, but a consequence of the absence of magnetic charge.

The way in which **H** is related to **B** and **M** is reviewed in Fig. 11.25, for both the SI and the Gaussian systems of units. These relations hold whether **M** is proportional to **B** or not. However, if **M** is proportional to **B**, then it will also be proportional to **H**. In fact, the traditional definition of the volume magnetic susceptibility χ_m is not the logically preferable one given in Eq. (11.52), but rather

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \qquad (\text{if } \mathbf{M} \propto \mathbf{B}), \tag{11.72}$$

which we shall reluctantly adopt from here on. If $\chi_m \ll 1$, which is commonly the case, there is negligible difference between the two definitions; see Exercise 11.38.

The permanent magnet in Fig. 11.22(b) is an instructive example of the relation of **H** to **B** and **M**. To obtain **H** at some point inside the magnetized material, we have to add vectorially to $\mathbf{B}/\mu_0$ at that point the vector $-\mathbf{M}$. Figure 11.26 depicts this for a particular point P. It turns out that the lines of **H** inside the magnet look just like the lines of **E** inside the polarized cylinder of Fig. 11.22(a). The reason for this is the following. In the permanent magnet there are no free currents at all. Consequently, the line integral of **H**, according to Eq. (11.70), must be zero around any closed path. You can see that this will be the case if the **H** lines look like the **E** lines in Fig. 11.22(a), for we know the line integral of that electrostatic field is zero around any closed path.

Said in a different way, if magnetic poles, rather than electric currents, really were the source of the magnetization, then the macroscopic magnetic field inside the magnetized material would look just like the macroscopic electric field inside the polarized material (because that field is produced by electric poles). The similarity of magnetic polarization and electric polarization would be complete. The **B** field in a (hypothetical) setup with magnetic poles looks like the **H** field in a (real) setup with current loops.

In the example of the permanent magnet, Eq. (11.72) does not apply. The magnetization vector **M** is not proportional to **H** but is determined, instead, by the previous treatment of the material. How this can come about will be explained in the following section.

For any material in which **M** is proportional to **H**, so that Eq. (11.72) applies as well as the basic relation, Eq. (11.68), we have

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}. \tag{11.73}$$

Hence **B** is proportional to **H**. The factor of proportionality, $\mu_0(1 + \chi_m)$, is called the *magnetic permeability* and denoted usually by μ:

$$\boxed{\mathbf{B} = \mu \mathbf{H}} \qquad \text{where} \quad \mu \equiv \mu_0(1 + \chi_m). \tag{11.74}$$

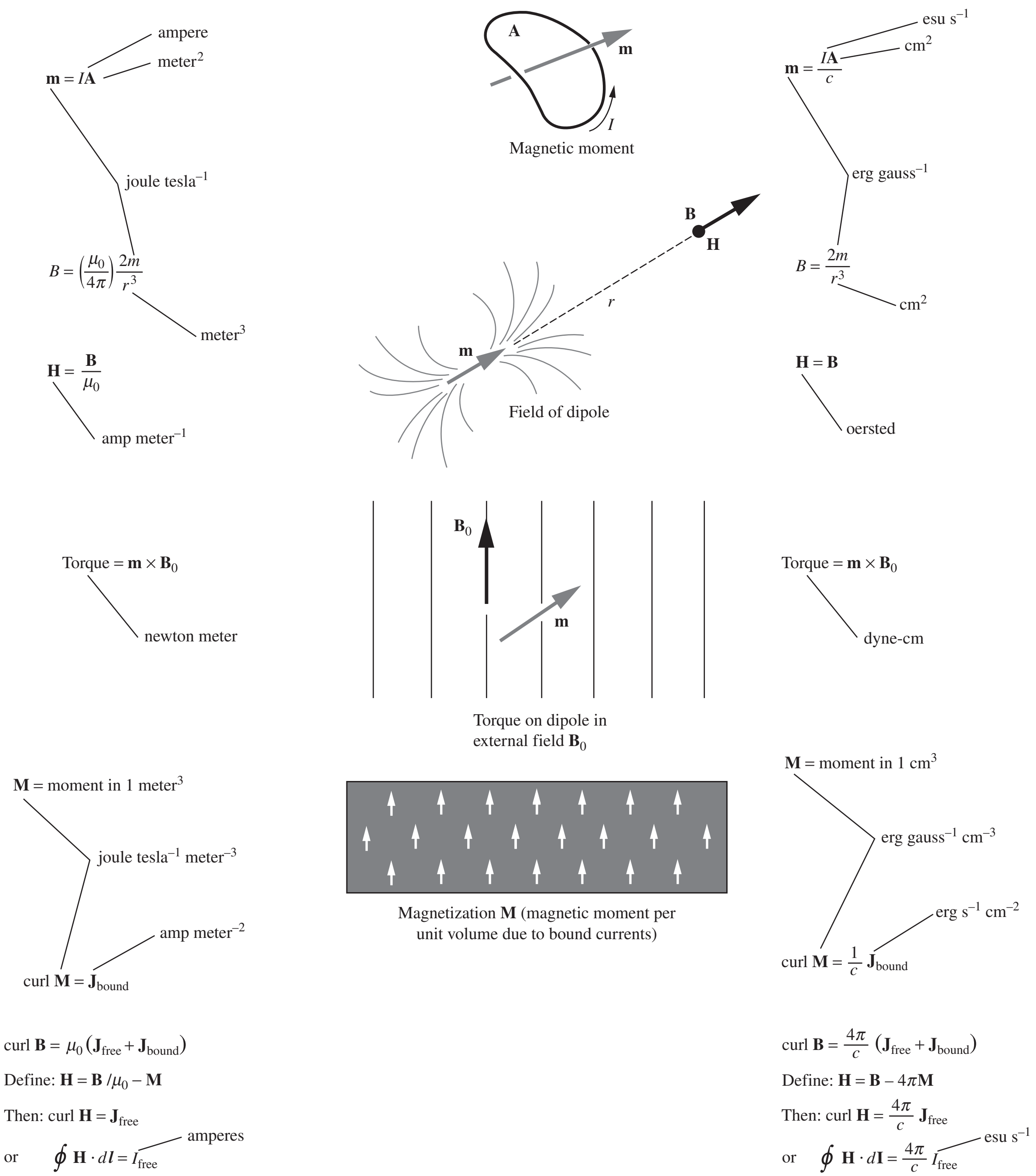


Figure 11.25.
Summary of relations involving $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, $\mathbf{m}$, $\mathbf{J}_{\text{free}}$, and $\mathbf{J}_{\text{bound}}$.

The permeability μ, rather than the susceptibility χ, is customarily used in describing ferromagnetism.

11.11 Ferromagnetism

Ferromagnetism has served and puzzled man for a long time. The *lodestone* (magnetite) was known in antiquity, and the influence on history of iron in the shape of compass needles was perhaps second only to that of iron in the shape of swords. For a century our electrical technology depended heavily on the circumstance that one abundant metal happens to possess this peculiar property. Nevertheless, it was only with the development of quantum mechanics that anything like a fundamental understanding of ferromagnetism was achieved.

We have already described some properties of ferromagnets. In a very strong magnetic field the force on a ferromagnetic substance is in such a direction as to pull it into a stronger field, as for paramagnetic materials, but instead of being proportional to the product of the field **B** and its gradient, the force is proportional to the gradient itself. As we remarked at the end of Section 11.4, this suggests that, if the field is strong enough, the magnetic moment acquired by the ferromagnet reaches some limiting magnitude. The direction of the magnetic moment vector must still be controlled by the field, for otherwise the force would not always act in the direction of increasing field intensity.

In permanent magnets we observe a magnetic moment even in the absence of any externally applied field, and it maintains its magnitude and direction even when external fields are applied, if they are not too strong. The field of the permanent magnet itself is always present, of course, and you may wonder whether it could not keep its own sources lined up. However, if you look again at Fig. 11.22(b) and Fig. 11.26, you will notice that **M** is generally not parallel to either **B** or **H**. This suggests that the magnetic dipoles must be clamped in direction by something other than purely magnetic forces.

The magnetization observed in ferromagnetic materials is much larger than we are used to in paramagnetic substances. Permanent magnets quite commonly have fields in the range of a few thousand gauss. A more characteristic quantity is the limiting value of the magnetization, the magnetic moment per unit volume, that the material acquires in a very strong field. This is called the *saturation* magnetization.

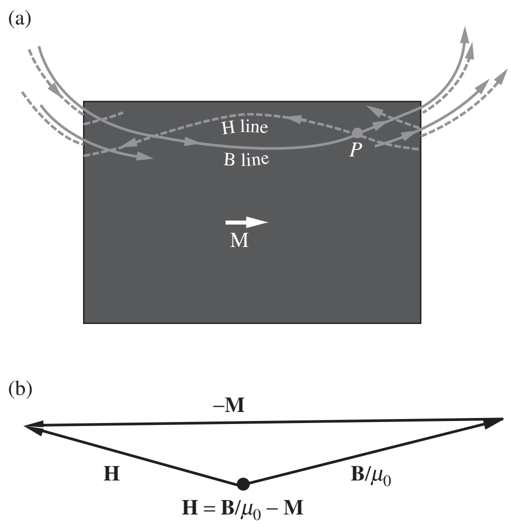


Figure 11.26.
(a) The relation of **B**, **H**, and **M** at a point inside the magnetized cylinder of Fig. 11.22(b).
(b) Relation of vectors at point P.

Example We can deduce the saturation magnetization of iron from the data in Table 11.1. In a field with a gradient of 17 tesla/m, the force on 1 kg of iron was 4000 newtons. From Eq. (11.20), which relates the force on a dipole to the field gradient, we find

$$m = \frac{F}{dB/dz} = \frac{4000 \text{ newtons}}{17 \text{ tesla/m}} = 235 \text{ joules/tesla} \quad \text{(for 1 kg)}. \tag{11.75}$$

To get the moment per cubic meter we multiply m by the density of iron, $7800\,\text{kg/m}^3$. The magnetization M is thus

$$\begin{aligned} M &= (235\ \text{joules/tesla-kg})(7800\ \text{kg/m}^3) \\ &= 1.83 \cdot 10^6\ \text{joules/(tesla-m}^3\text{)}. \end{aligned} \tag{11.76}$$

It is $\mu_0 M$, not M, that we should compare with field strengths in tesla. In the present case, $\mu_0 M$ has the value of 2.3 tesla.

It is more interesting to see how many electron spin moments this magnetization corresponds to. Dividing M by the electron moment given in Fig. 11.14, namely $9.3 \cdot 10^{-24}$ joule/tesla, we get about $2 \cdot 10^{29}$ spin moments per cubic meter. Now, 1 m^3 of iron contains about 10^{29} atoms. The limiting magnetization seems to correspond to about two lined-up spins per atom. As most of the electrons in the atom are paired off and have no magnetic effect at all, this indicates that we are dealing with substantially complete alignment of those few electron spins in the atom's structure that are at liberty to point in the same direction.

A very suggestive fact about ferromagnets is this: a given ferromagnetic substance, pure iron for example, loses its ferromagnetic properties quite abruptly if heated to a certain temperature. Above 770 °C, pure iron acts like a paramagnetic substance. Cooled below 770 °C, it immediately recovers its ferromagnetic properties. This transition temperature, called the *Curie point* after Pierre Curie who was one of its early investigators, is different for different substances. For pure nickel it is 358 °C.

What is this ferromagnetic behavior that so sharply distinguishes iron below 770 °C from iron above 770 °C, and from copper at any temperature? It is the *spontaneous* lining up in one direction of the atomic magnetic moments, which implies alignment of the spin axes of certain electrons in each iron atom. By *spontaneous,* we mean that no external magnetic field need be involved. Over a region in the iron large enough to contain millions of atoms, the spins and magnetic moments of nearly all the atoms are pointing in the same direction. Well below the Curie point – at room temperature, for instance, in the case of iron – the alignment is nearly perfect. If you could magically look into the interior of a crystal of metallic iron and see the elementary magnetic moments as vectors with arrowheads on them, you might see something like Fig. 11.27.

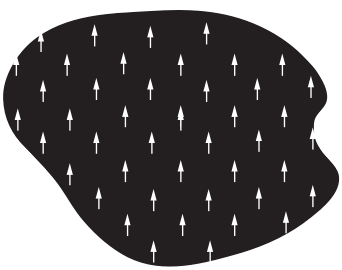


Figure 11.27.
The orderliness of the spin directions in a small region in a crystal of iron. Each arrow represents the magnetic moment of one iron atom.

It is hardly surprising that a high temperature should destroy this neat arrangement. Thermal energy is the enemy of order, so to speak. A crystal, an orderly arrangement of atoms, changes to a liquid, a much less orderly arrangement, at a sharply defined temperature, the melting point. The melting point, like the Curie point, is different for different substances. Let us concentrate here on the ordered state itself. Two or three questions are obvious.

Question 1 What makes the spins line up and keeps them lined up?

Question 2 How, if there is no external field present, can the spins choose one direction rather than another? Why didn't all the moments in Fig. 11.27 point down, or to the right, or to the left?

Question 3 If the atomic moments *are* all lined up, why isn't every piece of iron at room temperature a strong magnet?

The answers to these three questions will help us to understand, in a general way at least, the behavior of ferromagnetic materials when an external field, neither very strong nor very weak, is applied. That includes a very rich variety of phenomena that we haven't even described yet.

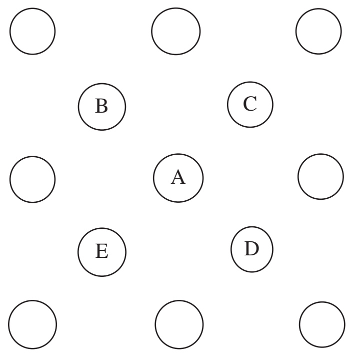


Figure 11.28.
An atom A and its nearest neighbors in the crystal lattice. (Of course, the lattice is really three-dimensional.)

Answer 1 For some reason connected with the quantum mechanics of the structure of the iron atom, it is energetically favorable for the spins of adjacent iron atoms to be parallel. This is *not* due to their magnetic interaction. It is a stronger effect than that, and moreover it favors parallel spins whether like this ↑↑ or like this →→ (dipole interactions don't work that way – see Exercise 10.29). Now if atom A (Fig. 11.28) wants to have its spin in the same direction as that of its neighbors, atoms B, C, D, and E, and each of *them* prefers to have its spin in the same direction as the spin of *its* neighbors, including atom A, you can readily imagine that if a local majority ever develops there will be a strong tendency to "make it unanimous," and then the fad will spread.

Answer 2 Accident somehow determines which of the various equivalent directions in the crystal is chosen, if we commence from a disordered state – as, for example, if the iron is cooled through its Curie point without any external field applied. Pure iron consists of body-centered cubic crystals. Each atom has eight nearest neighbors. The symmetry of the environment imposes itself on every physical aspect of the atom, including the coupling between spins. In iron the cubic axes happen to be the axes of easiest magnetization. That is, the spins like to point in the same direction, but they like it even better if that direction is one of the six directions $\pm\hat{\mathbf{x}}$, $\pm\hat{\mathbf{y}}$, $\pm\hat{\mathbf{z}}$ (Fig. 11.29). This is important because it means that the spins cannot easily swivel around *en masse* from one of the easy directions to an equivalent one at right angles. To do so, they would have to swing through *less* favorable orientations on the way. It is just this hindrance that makes permanent magnets possible.

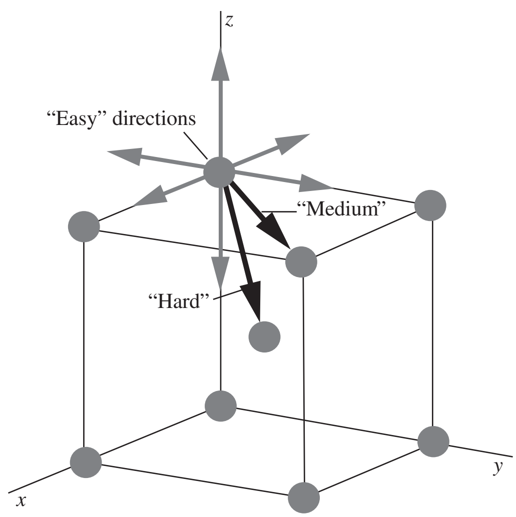


Figure 11.29.
In iron the energetically preferred direction of magnetization is along a cubic axis of the crystal.

Answer 3 An apparently unmagnetized piece of iron is actually composed of many *domains,* in each of which the spins are all lined up one way, but in a direction different from that of the spins in neighboring domains. On the average over the whole piece of "unmagnetized" iron, all directions are equally represented, so no large-scale magnetic field results. Even in a single crystal the magnetic domains establish themselves. The domains are usually microscopic in the everyday sense of the

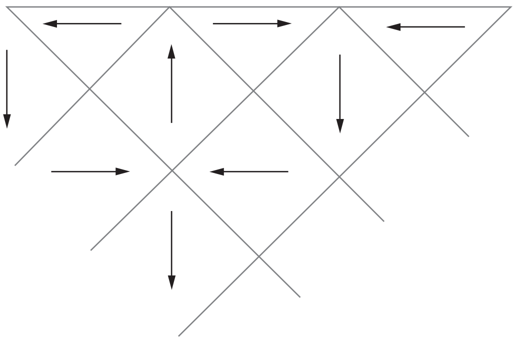


Figure 11.30.
Possible arrangement of magnetic domains in a single uniform crystal of iron.

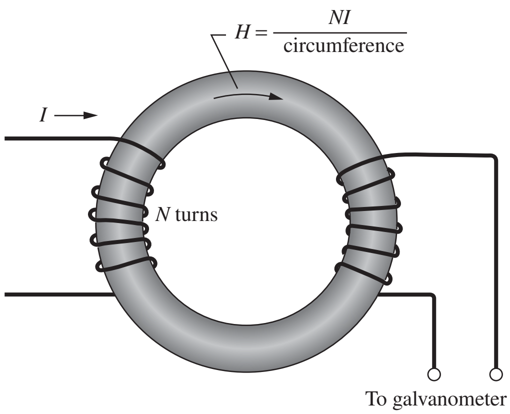


Figure 11.31.
Arrangement for investigating the relation between **B** and **M**, or **B** and **H**, in a ferromagnetic material.

word. In fact, they can be made visible under a low-power microscope. That is still enormous, of course, on an atomic scale, so a magnetic domain typically includes billions of elementary magnetic moments. Figure 11.30 depicts a division into domains. The division comes about because it is cheaper in energy than an arrangement with all the spins pointing in one direction. The latter arrangement would be a permanent magnet with a strong field extending out into the space around it. The energy stored in this exterior field is larger than the energy needed to turn some small fraction of the spins in the crystal, namely those at a domain boundary, out of line with their immediate neighbors. The domain structure is thus the outcome of an energy-minimization contest.

If we wind a coil of wire around an iron rod, we can apply a magnetic field to the material by passing a current through the wire. In this field, moments pointing parallel to the field will have a lower energy than those pointing antiparallel, or in some other direction. This favors some domains over others; those that happen to have a favorably oriented moment direction[7] will tend to grow at the expense of the others, if that is possible. A domain grows like a club, that is, by expanding its membership. This happens at the boundaries. Spins belonging to an unfavored domain, but located next to the boundary with a favored domain, simply switch allegiance by adopting the favored direction. That merely shifts the domain boundary, which is nothing more than the dividing surface between the two classes of spins. This happens rather easily in single crystals. That is, a very weak applied field can bring about, through boundary movement, a very large domain growth, and hence a large overall change in magnetization. Depending on the grain structure of the material, however, the movement of domain boundaries can be difficult.

If the applied field does not happen to lie along one of the "easy" directions (in the case of a cubic crystal, for example), the exhaustion of the unfavored domains still leaves the moments not pointing exactly parallel to the field. It may now take a considerably stronger field to pull them into line with the field direction so as to create, finally, the maximum magnetization possible.

Let us look at the large-scale consequences of this, as they appear in the magnetic behavior of a piece of iron under various applied fields. A convenient experimental arrangement is an iron torus, around which we wind two coils (Fig. 11.31). This affords a practically uniform field within the iron, with no end effects to complicate matters. By measuring

[7] We tend to use *spins* and *moments* almost interchangeably in this discussion. The moment is an intrinsic aspect of the spin, and if one is lined up so is the other. To be meticulous, we should remind the reader that in the case of the electron the magnetic moment and angular momentum vectors point in opposite directions (Fig. 11.14).

the voltage induced in one of the coils, we can determine changes in flux Φ, and hence in $\mathbf{B}$ inside the iron. If we keep track of the *changes* in $\mathbf{B}$, starting from $B = 0$, we always know what $\mathbf{B}$ is. A current through the other coil establishes $\mathbf{H}$, which we take as the independent variable. If we know $\mathbf{B}$ and $\mathbf{H}$, we can always compute $\mathbf{M}$. It is more usual to plot $\mathbf{B}$ rather than $\mathbf{M}$, as a function of $\mathbf{H}$. A typical B-H curve for iron is shown in Fig. 11.32. Note the different units on the axes; B is measured in tesla while H is measured in amps/meter. If there were no iron in the coil, $\mathbf{B}$ would equal $\mu_0\mathbf{H}$, so $H = 1$ amp/meter would be worth exactly $B = 4\pi \cdot 10^{-7}$ tesla. Or equivalently, $H = 300$ amps/meter would yield $B \approx 4 \cdot 10^{-4}$ tesla. But with the iron present, the resulting B field is much larger. We see from the figure that when $H = 300$ amps/meter, B has risen to more than 1 tesla. Of course, B and H here refer to an average throughout the whole iron ring; the fine domain structure as such never exhibits itself.

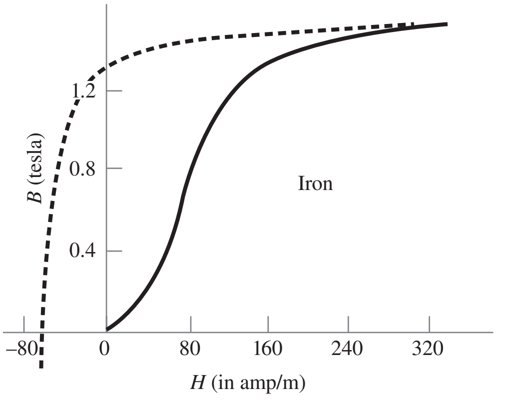


Figure 11.32.
Magnetization curve for fairly pure iron. The dashed curve is obtained as H is reduced from a high positive value.

Starting with unmagnetized iron, $B=0$ and $H=0$, increasing H causes B to rise in a conspicuously nonlinear way, slowly at first, then more rapidly, then very slowly, finally flattening off. What actually becomes constant in the limit is not B but M. In this graph, however, since $\mathbf{M}=\mathbf{B}/\mu_0-\mathbf{H}$, and $H \ll B/\mu_0$, the difference between B and $\mu_0 M$ is not appreciable.

The lower part of the B-H curve is governed by the motion of domain boundaries, that is, by the growth of "right-pointing" domains at the expense of "wrong-pointing" domains. In the upper flattening part of the curve, the atomic moments are being pulled by "brute force" into line with the field. The iron here is an ordinary polycrystalline metal, so only a small fraction of the microcrystals will be fortunate enough to have an easy direction lined up with the field direction.

If we now slowly decrease the current in the coil, thus lowering H, the curve *does not retrace itself*. Instead, we find the behavior given by the dashed curve in Fig. 11.32. This irreversibility is called *hysteresis*. It is largely due to the domain boundary movements being partially irreversible. The reasons are not obvious from anything we have said, but are well understood by physicists who work on ferromagnetism. The irreversibility is a nuisance, and a cause of energy loss in many technical applications of ferromagnetic materials – for instance, in alternating-current transformers. But it is indispensable for permanent magnetization, and for such applications, one wants to enhance the irreversibility. Figure 11.33 shows the corresponding portion of the B-H curve for a good permanent magnet alloy. Note that H has to become about 50,000 amps/meter in the *reverse* direction before B is reduced to zero. If the coil is simply switched off and removed, we are left with B at 1.3 tesla, called the *remanence*. Since H is zero, this is essentially the same as $\mu_0 M$. The alloy has acquired a permanent magnetization, that is, one that will persist indefinitely if it is exposed only to weak magnetic fields.

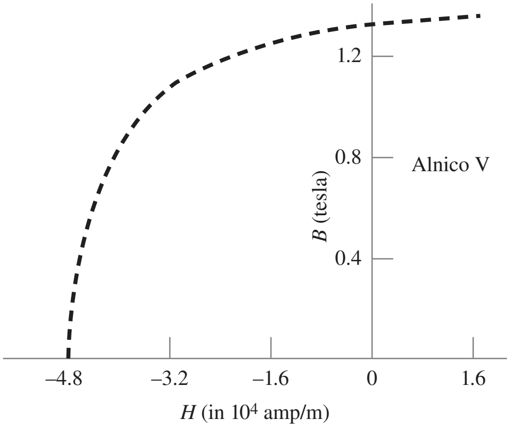


Figure 11.33.
Alnico V is an alloy of aluminum, nickel, and cobalt that is used for permanent magnets. Compare this portion of its magnetization curve with the corresponding portion of the characteristic for a "soft" magnetic material, shown in Fig. 11.32.

All the information that is stored on magnetic tapes and disks owes its permanence to this physical phenomenon.

6.3 ■ THE AUXILIARY FIELD H

6.3.1 ■ Ampère's Law in Magnetized Materials

In Sect. 6.2, we found that the effect of magnetization is to establish bound currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ within the material and $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$ on the surface. The field due to magnetization of the medium is just the field produced by these bound currents. We are now ready to put everything together: the field attributable to bound currents, plus the field due to everything else—which I shall call the **free current**. The free current might flow through wires imbedded in the magnetized substance or, if the latter is a conductor, through the material itself. In any event, the total current can be written as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f. \tag{6.17}$$

There is no new physics in Eq. 6.17; it is simply a *convenience* to separate the current into these two parts, because they got there by quite different means: the free current is there because somebody hooked up a wire to a battery—it involves actual transport of charge; the bound current is there because of magnetization—it results from the conspiracy of many aligned atomic dipoles.

In view of Eqs. 6.13 and 6.17, Ampère's law can be written

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M}),$$

or, collecting together the two curls:

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f.$$

The quantity in parentheses is designated by the letter **H**:

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.} \tag{6.18}$$

In terms of **H**, then, Ampère's law reads

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f,} \tag{6.19}$$

or, in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}, \tag{6.20}$$

where $I_{f_{\text{enc}}}$ is the total *free* current passing through the Amperian loop.

H plays a role in magnetostatics analogous to **D** in electrostatics: Just as **D** allowed us to write *Gauss's* law in terms of the free *charge* alone, **H** permits us to express *Ampère's* law in terms of the free *current* alone—and free current is what we control directly. Bound current, like bound charge, comes along for the ride—the material gets magnetized, and this results in bound currents; we cannot turn them on or off independently, as we can free currents. In applying Eq. 6.20, all we need to worry about is the *free* current, which we know about because we *put* it there. In particular, when symmetry permits, we can calculate **H** immediately from Eq. 6.20 by the usual Ampère's law methods. (For example, Probs. 6.7 and 6.8 can be done in one line by noting that $\mathbf{H} = \mathbf{0}$.)

Example 6.2. A long copper rod of radius R carries a uniformly distributed (free) current I (Fig. 6.19). Find **H** inside and outside the rod.

Solution

Copper is weakly diamagnetic, so the dipoles will line up opposite to the field. This results in a bound current running antiparallel to I, within the wire, and parallel to I along the surface (Fig. 6.20). Just how great these bound currents will

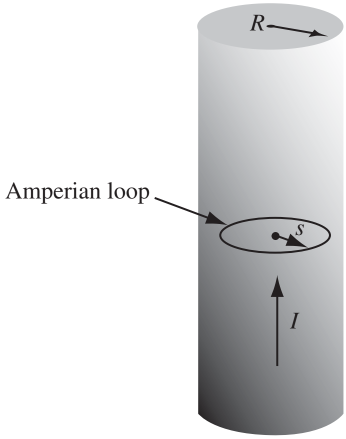


FIGURE 6.19

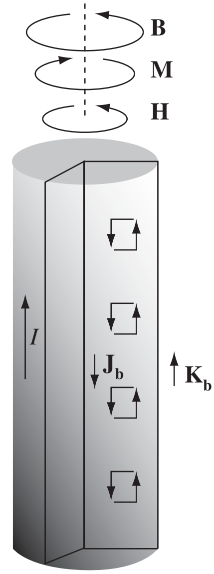


FIGURE 6.20

be we are not yet in a position to say—but in order to calculate **H**, it is sufficient to realize that all the currents are longitudinal, so **B**, **M**, and therefore also **H**, are circumferential. Applying Eq. 6.20 to an Amperian loop of radius $s < R$,

$$H(2\pi s) = I_{f_{\text{enc}}} = I\frac{\pi s^2}{\pi R^2},$$

so, inside the wire,

$$\mathbf{H} = \frac{I}{2\pi R^2} s\,\hat{\boldsymbol{\phi}} \qquad (s \le R). \tag{6.21}$$

Outside the wire

$$\mathbf{H} = \frac{I}{2\pi s}\,\hat{\boldsymbol{\phi}} \qquad (s \ge R). \tag{6.22}$$

In the latter region (as always, in empty space) $\mathbf{M} = \mathbf{0}$, so

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}} \qquad (s \ge R),$$

the same as for a *non*magnetized wire (Ex. 5.7). *Inside* the wire **B** cannot be determined at this stage, since we have no way of knowing **M** (though in practice the magnetization in copper is so slight that for most purposes we can ignore it altogether).

As it turns out, **H** is a more useful quantity than **D**. In the laboratory, you will frequently hear people talking about **H** (more often even than **B**), but you will never hear anyone speak of **D** (only **E**). The reason is this: To build an electromagnet you run a certain (free) current through a coil. The *current* is the thing you read on the dial, and this determines **H** (or at any rate, the line integral of **H**); **B** depends on the specific materials you used and even, if iron is present, on the history of your magnet. On the other hand, if you want to set up an *electric* field, you do *not* plaster a known free charge on the plates of a parallel plate capacitor; rather, you connect them to a battery of known *voltage*. It's the *potential difference* you read on your dial, and that determines **E** (or rather, the line integral of **E**); **D** depends on the details of the dielectric you're using. If it were easy to measure charge, and hard to measure potential, then you'd find experimentalists talking about **D** instead of **E**. So the relative familiarity of **H**, as contrasted with **D**, derives from purely practical considerations; theoretically, they're on an equal footing.

Many authors call **H**, not **B**, the "magnetic field." Then they have to invent a new word for **B**: the "flux density," or magnetic "induction" (an absurd choice, since that term already has at least two other meanings in electrodynamics). Anyway, **B** is indisputably the fundamental quantity, so I shall continue to call it the "magnetic field," as everyone does in the spoken language. **H** has no sensible name: just call it "**H**."[7]

Problem 6.12 An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis,

$$\mathbf{M} = ks\,\hat{\mathbf{z}},$$

where k is a constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) As in Sect. 6.2, locate all the bound currents, and calculate the field they produce.

(b) Use Ampère's law (in the form of Eq. 6.20) to find **H**, and then get **B** from Eq. 6.18. (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

Problem 6.13 Suppose the field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = (1/\mu_0)\mathbf{B}_0 - \mathbf{M}$, where **M** is a "frozen-in" magnetization.

(a) Now a small spherical cavity is hollowed out of the material (Fig. 6.21). Find the field at the center of the cavity, in terms of $\mathbf{B}_0$ and **M**. Also find **H** at the center of the cavity, in terms of $\mathbf{H}_0$ and **M**.

(b) Do the same for a long needle-shaped cavity running parallel to **M**.

(c) Do the same for a thin wafer-shaped cavity perpendicular to **M**.

[7] For those who disagree, I quote A. Sommerfeld's *Electrodynamics* (New York: Academic Press, 1952), p. 45: "The unhappy term 'magnetic field' for **H** should be avoided as far as possible. It seems to us that this term has led into error none less than Maxwell himself . . ."

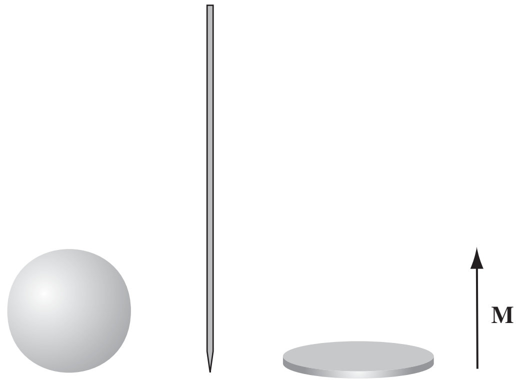


FIGURE 6.21

Assume the cavities are small enough so $\mathbf{M}$, $\mathbf{B}_0$, and $\mathbf{H}_0$ are essentially constant. Compare Prob. 4.16. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite magnetization.]

6.3.2 ■ A Deceptive Parallel

Equation 6.19 looks just like Ampère's original law (Eq. 5.56), except that the *total* current is replaced by the *free* current, and $\mathbf{B}$ is replaced by $\mu_0\mathbf{H}$. As in the case of $\mathbf{D}$, however, I must warn you against reading too much into this correspondence. It does *not* say that $\mu_0\mathbf{H}$ is "just like $\mathbf{B}$, only its source is $\mathbf{J}_f$ instead of $\mathbf{J}$." For the curl alone does not determine a vector field—you must *also* know the divergence. And whereas $\nabla \cdot \mathbf{B} = 0$, the divergence of $\mathbf{H}$ is *not*, in general, zero. In fact, from Eq. 6.18

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}. \tag{6.23}$$

Only when the divergence of $\mathbf{M}$ vanishes is the parallel between $\mathbf{B}$ and $\mu_0\mathbf{H}$ faithful.

If you think I'm being pedantic, consider the example of the bar magnet—a short cylinder of iron that carries a permanent uniform magnetization $\mathbf{M}$ parallel to its axis. (See Probs. 6.9 and 6.14.) In this case there is no free current anywhere, and a naïve application of Eq. 6.20 might lead you to suppose that $\mathbf{H} = \mathbf{0}$, and hence that $\mathbf{B} = \mu_0\mathbf{M}$ inside the magnet and $\mathbf{B} = \mathbf{0}$ outside, which is nonsense. It is quite true that the *curl* of $\mathbf{H}$ vanishes everywhere, but the divergence does not. (Can you see where $\nabla \cdot \mathbf{M} \neq 0$?) *Advice*: When you are asked to find $\mathbf{B}$ or $\mathbf{H}$ in a problem involving magnetic materials, first look for symmetry. If the problem exhibits cylindrical, plane, solenoidal, or toroidal symmetry, then you can get $\mathbf{H}$ directly from Eq. 6.20 by the usual Ampère's law methods. (Evidently, in such cases $\nabla \cdot \mathbf{M}$ is automatically zero, since the free current alone determines the answer.) If the requisite symmetry is absent, you'll have to think of another approach, and in particular you must *not* assume that **H** is zero just because there is no free current in sight.

6.3.3 ■ Boundary Conditions

The magnetostatic boundary conditions of Sect. 5.4.2 can be rewritten in terms of **H** and the *free* current. From Eq. 6.23 it follows that

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}), \tag{6.24}$$

while Eq. 6.19 says

$$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}. \tag{6.25}$$

In the presence of materials, these are sometimes more useful than the corresponding boundary conditions on **B** (Eqs. 5.74 and 5.76):

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0, \tag{6.26}$$

and

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}). \tag{6.27}$$

You might want to check them, for Ex. 6.2 or Prob. 6.14.

Problem 6.14 For the bar magnet of Prob. 6.9, make careful sketches of **M**, **B**, and **H**, assuming L is about $2a$. Compare Prob. 4.17.

Problem 6.15 If $\mathbf{J}_f = \mathbf{0}$ everywhere, the curl of **H** vanishes (Eq. 6.19), and we can express **H** as the gradient of a scalar potential W:

$$\mathbf{H} = -\nabla W.$$

According to Eq. 6.23, then,

$$\nabla^2 W = (\nabla \cdot \mathbf{M}),$$

so W obeys Poisson's equation, with $\nabla \cdot \mathbf{M}$ as the "source." This opens up all the machinery of Chapter 3. As an example, find the field inside a uniformly magnetized sphere (Ex. 6.1) by separation of variables. [*Hint:* $\nabla \cdot \mathbf{M} = 0$ everywhere except at the surface ($r = R$), so W satisfies Laplace's equation in the regions $r < R$ and $r > R$; use Eq. 3.65, and from Eq. 6.24 figure out the appropriate boundary condition on W.]

6.4 ■ LINEAR AND NONLINEAR MEDIA

6.4.1 ■ Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when **B** is removed, **M** disappears. In fact, for most substances the magnetization is *proportional* to the field, provided the field is not too strong. For notational consistency with the electrical case (Eq. 4.30), I *should* express the proportionality thus:

$$\mathbf{M} = \frac{1}{\mu_0}\chi_m \mathbf{B} \quad \text{(incorrect!)}. \tag{6.28}$$

But custom dictates that it be written in terms of **H**, instead of **B**:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}.} \tag{6.29}$$

The constant of proportionality χ_m is called the **magnetic susceptibility**; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5} (see Table 6.1).

Materials that obey Eq. 6.29 are called **linear media**. In view of Eq. 6.18,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}, \tag{6.30}$$

for linear media. Thus **B** is *also* proportional to **H**:[8]

$$\mathbf{B} = \mu \mathbf{H}, \tag{6.31}$$

where

$$\mu \equiv \mu_0(1 + \chi_m). \tag{6.32}$$

μ is called the **permeability** of the material.[9] In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the **permeability of free space**.

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.7×10^{-4}	Oxygen (O_2)	1.7×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.2×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.0×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.7×10^{-4}
Carbon Dioxide	-1.1×10^{-8}	Liquid Oxygen (−200° C)	3.9×10^{-3}
Hydrogen (H_2)	-2.1×10^{-9}	Gadolinium	4.8×10^{-1}

TABLE 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, Inc., 2010) and other references.

[8]Physically, therefore, Eq. 6.28 would say exactly the same as Eq. 6.29, only the constant χ_m would have a different value. Equation 6.29 is a little more convenient, because experimentalists find it handier to work with **H** than **B**.

[9]If you factor out μ_0, what's left is called the **relative permeability**: $\mu_r \equiv 1 + \chi_m = \mu/\mu_0$. By the way, formulas for **H** in terms of **B** (Eq. 6.31, in the case of linear media) are called **constitutive relations**, just like those for **D** in terms of **E**.

Example 6.3. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

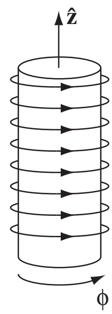


FIGURE 6.22

Solution

Since **B** is due in part to bound currents (which we don't yet know), we cannot compute it directly. However, this is one of those symmetrical cases in which we can get **H** from the free current alone, using Ampère's law in the form of Eq. 6.20:

$$\mathbf{H} = nI\,\hat{\mathbf{z}}$$

(Fig. 6.22). According to Eq. 6.31, then,

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}.$$

If the medium is paramagnetic, the field is slightly enhanced; if it's diamagnetic, the field is somewhat reduced. This reflects the fact that the bound surface current

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m(\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m nI\,\hat{\boldsymbol{\phi}}$$

is in the same direction as I, in the former case ($\chi_m > 0$), and opposite in the latter ($\chi_m < 0$).

You might suppose that linear media escape the defect in the parallel between **B** and **H**: since **M** and **H** are now proportional to **B**, does it not follow that their divergence, like **B**'s, must always vanish? Unfortunately, it does *not*;[10] at the *boundary* between two materials of different permeability, the divergence of **M** can actually be infinite. For instance, at the end of a cylinder of linear paramagnetic material, **M** is zero on one side but not on the other. For the "Gaussian pillbox" shown in Fig. 6.23, $\oint \mathbf{M} \cdot d\mathbf{a} \neq 0$, and hence, by the divergence theorem, $\boldsymbol{\nabla} \cdot \mathbf{M}$ cannot vanish everywhere within it.

[10]Formally, $\boldsymbol{\nabla} \cdot \mathbf{H} = \boldsymbol{\nabla} \cdot \left(\frac{1}{\mu}\mathbf{B}\right) = \frac{1}{\mu}\boldsymbol{\nabla} \cdot \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla}\left(\frac{1}{\mu}\right) = \mathbf{B} \cdot \boldsymbol{\nabla}\left(\frac{1}{\mu}\right)$, so **H** is *not* divergenceless (in general) at points where μ is changing.

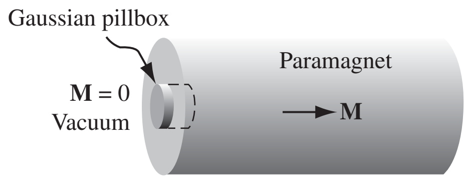


FIGURE 6.23

Incidentally, the volume bound current density in a homogeneous linear material is proportional to the *free* current density:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f. \tag{6.33}$$

In particular, unless free current actually flows *through* the material, all bound current will be at the surface.

Problem 6.16 A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface (Fig. 6.24). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

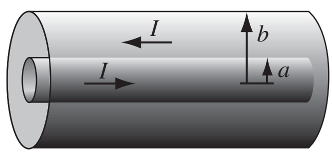


FIGURE 6.24

Problem 6.17 A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the *net* bound current flowing down the wire?

! **Problem 6.18** A sphere of linear magnetic material is placed in an otherwise uniform magnetic field $\mathbf{B}_0$. Find the new field inside the sphere. [*Hint:* See Prob. 6.15 or Prob. 4.23.]

Problem 6.19 On the basis of the naïve model presented in Sect. 6.1.3, estimate the magnetic susceptibility of a diamagnetic metal such as copper. Compare your answer with the empirical value in Table 6.1, and comment on any discrepancy.

6.4.2 ■ Ferromagnetism

In a linear medium, the alignment of atomic dipoles is maintained by a magnetic field imposed from the outside. Ferromagnets—which are emphatically *not* linear[11]—require no external fields to sustain the magnetization; the alignment is "frozen in." Like paramagnetism, ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons. The new feature, which makes ferromagnetism so different from paramagnetism, is the interaction between nearby dipoles: In a ferromagnet, *each dipole "likes" to point in the same direction as its neighbors*. The *reason* for this preference is essentially quantum mechanical, and I shall not endeavor to explain it here; it is enough to know that the correlation is so strong as to align virtually 100% of the unpaired electron spins. If you could somehow magnify a piece of iron and "see" the individual dipoles as tiny arrows, it would look something like Fig. 6.25, with all the spins pointing the same way.

But if that is true, why isn't every wrench and nail a powerful magnet? The answer is that the alignment occurs in relatively small patches, called **domains**. Each domain contains billions of dipoles, all lined up (these domains are actually *visible* under a microscope, using suitable etching techniques—see Fig. 6.26), but the domains *themselves* are randomly oriented. The household wrench contains an enormous number of domains, and their magnetic fields cancel, so the wrench as a whole is not magnetized. (Actually, the orientation of domains is not *completely* random; within a given crystal, there may be some preferential alignment along the crystal axes. But there will be just as many domains pointing one way as the other, so there is still no large-scale magnetization. Moreover, the crystals themselves are randomly oriented within any sizable chunk of metal.)

How, then, would you produce a **permanent magnet**, such as they sell in toy stores? If you put a piece of iron into a strong magnetic field, the torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ tends to align the dipoles parallel to the field. Since they like to stay parallel to their neighbors, most of the dipoles will resist this torque. However,

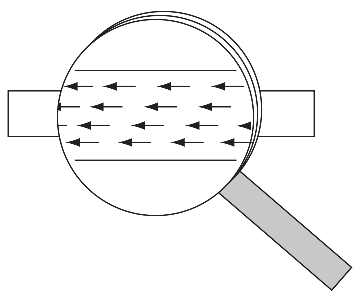
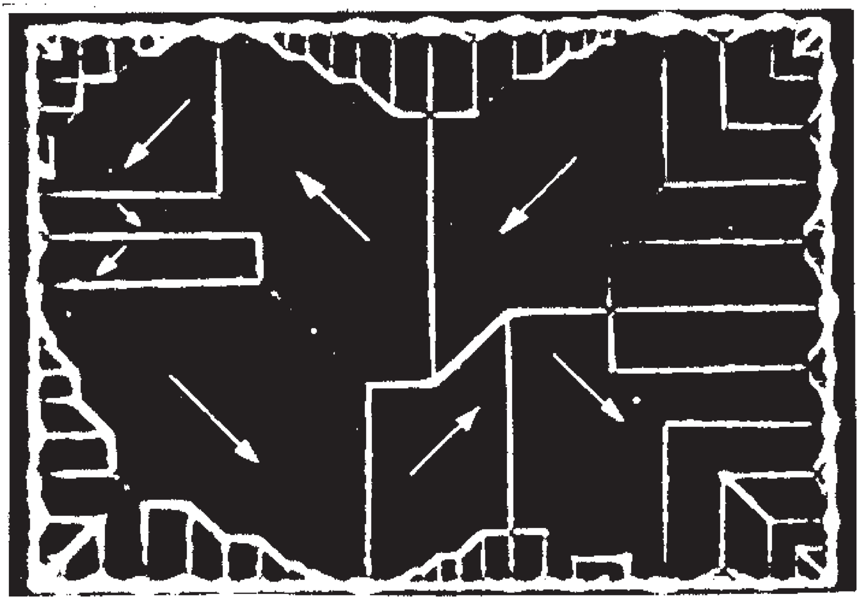


FIGURE 6.25

[11]In this sense, it is misleading to speak of the susceptibility or permeability of a ferromagnet. The terms *are* used for such materials, but they refer to the proportionality factor between a *differential* increase in $\mathbf{H}$ and the resulting *differential* change in $\mathbf{M}$ (or $\mathbf{B}$); moreover, they are not constants, but functions of $\mathbf{H}$.



Ferromagnetic domains. (Photo courtesy of R. W. DeBlois)

FIGURE 6.26

at the *boundary* between two domains, there are *competing* neighbors, and the torque will throw its weight on the side of the domain most nearly parallel to the field; this domain will win some converts, at the expense of the less favorably oriented one. The net effect of the magnetic field, then, is to *move the domain boundaries*. Domains parallel to the field grow, and the others shrink. If the field is strong enough, one domain takes over entirely, and the iron is said to be **saturated**.

It turns out that this process (the shifting of domain boundaries in response to an external field) is not entirely reversible: When the field is switched off, there will be *some* return to randomly oriented domains, but it is far from complete—there remains a preponderance of domains in the original direction. You now have a permanent magnet.

A simple way to accomplish this, in practice, is to wrap a coil of wire around the object to be magnetized (Fig. 6.27). Run a current I through the coil; this provides the external magnetic field (pointing to the left in the diagram). As you increase the current, the field increases, the domain boundaries move, and the magnetization grows. Eventually, you reach the saturation point, with all the dipoles aligned, and a further increase in current has no effect on $\mathbf{M}$ (Fig. 6.28, point b).

Now suppose you *reduce* the current. Instead of retracing the path back to $M = 0$, there is only a *partial* return to randomly oriented domains; M decreases, but even with the current off there is some residual magnetization (point c). The wrench is now a permanent magnet. If you want to eliminate the remaining magnetization, you'll have to run a current backwards through the coil (a negative I). Now the external field points to the right, and as you increase I (negatively),

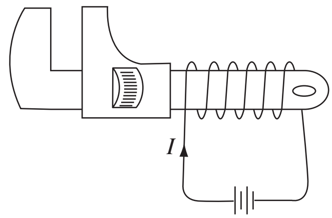


FIGURE 6.27

M drops down to zero (point d). If you turn I still higher, you soon reach saturation in the other direction—all the dipoles now pointing to the *right* (e). At this stage, switching off the current will leave the wrench with a permanent magnetization to the right (point f). To complete the story, turn I on again in the positive sense: M returns to zero (point g), and eventually to the forward saturation point (b).

The path we have traced out is called a **hysteresis loop**. Notice that the magnetization of the wrench depends not only on the applied field (that is, on I), but also on its previous magnetic "history."[12] For instance, at three different times in our experiment the current was zero (a, c, and f), yet the magnetization was different for each of them. Actually, it is customary to draw hysteresis loops as plots of B against H, rather than M against I. (If our coil is approximated by a long solenoid, with n turns per unit length, then $H = nI$, so H and I are proportional. Meanwhile, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, but in practice M is huge compared to H, so to all intents and purposes $\mathbf{B}$ is proportional to $\mathbf{M}$.)

To make the units consistent (teslas), I have plotted $(\mu_0 H)$ horizontally (Fig. 6.29); notice, however, that the vertical scale is 10^4 times greater than the horizontal one. Roughly speaking, $\mu_0\mathbf{H}$ is the field our coil *would* have produced in the absence of any iron; $\mathbf{B}$ is what we *actually* got, and compared to $\mu_0\mathbf{H}$, it is gigantic. A little current goes a long way, when you have ferromagnetic materials

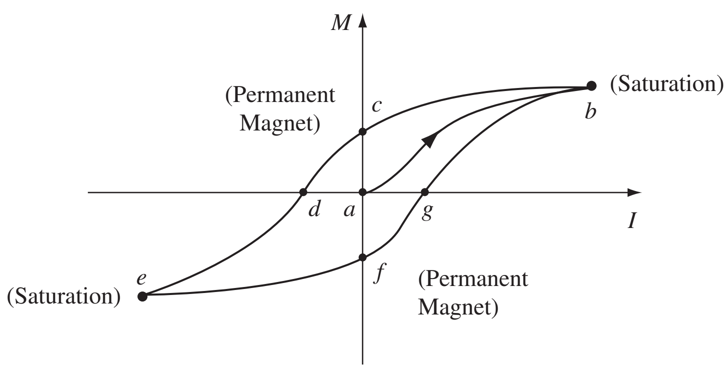


FIGURE 6.28

[12]Etymologically, the word *hysteresis* has nothing to do with the word *history*—nor with the word *hysteria*. It derives from a Greek verb meaning "lag behind."

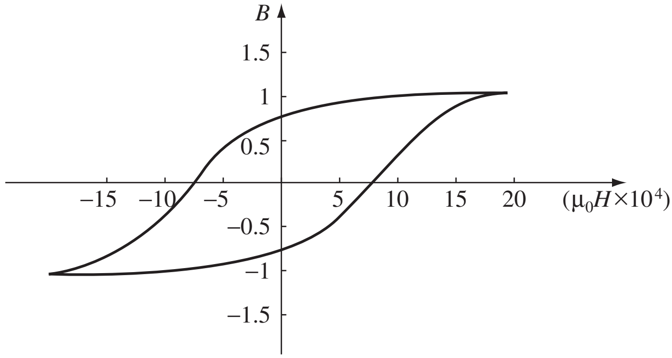


FIGURE 6.29

around. That's why anyone who wants to make a powerful electromagnet will wrap the coil around an iron core. It doesn't take much of an external field to move the domain boundaries, and when you do that, you have all the dipoles in the iron working with you.

One final point about ferromagnetism: It all follows, remember, from the fact that the dipoles within a given domain line up parallel to one another. Random thermal motions compete with this ordering, but as long as the temperature doesn't get too high, they cannot budge the dipoles out of line. It's not surprising, though, that *very* high temperatures do destroy the alignment. What *is* surprising is that this occurs at a precise temperature (770° C, for iron). Below this temperature (called the **Curie point**), iron is ferromagnetic; above, it is paramagnetic. The Curie point is rather like the boiling point or the freezing point in that there is no *gradual* transition from ferro- to para-magnetic behavior, any more than there is between water and ice. These abrupt changes in the properties of a substance, occurring at sharply defined temperatures, are known in statistical mechanics as **phase transitions**.

Problem 6.20 How would you go about *de*magnetizing a permanent magnet (such as the wrench we have been discussing, at point c in the hysteresis loop)? That is, how could you restore it to its original state, with $M = 0$ at $I = 0$?

Problem 6.21

(a) Show that the energy of a magnetic dipole in a magnetic field $\mathbf{B}$ is

$$\boxed{U = -\mathbf{m} \cdot \mathbf{B}.} \tag{6.34}$$

[Assume that the *magnitude* of the dipole moment is fixed, and all you have to do is move it into place and rotate it into its final orientation. The energy required to keep the current flowing is a different problem, which we will confront in Chapter 7.] Compare Eq. 4.6.

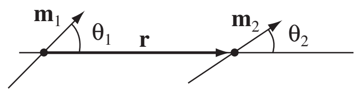


FIGURE 6.30

(b) Show that the interaction energy of two magnetic dipoles separated by a displacement **r** is given by

$$U = \frac{\mu_0}{4\pi}\frac{1}{r^3}[\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})]. \tag{6.35}$$

Compare Eq. 4.7.

(c) Express your answer to (b) in terms of the angles θ_1 and θ_2 in Fig. 6.30, and use the result to find the stable configuration two dipoles would adopt if held a fixed distance apart, but left free to rotate.

(d) Suppose you had a large collection of compass needles, mounted on pins at regular intervals along a straight line. How would they point (assuming the earth's magnetic field can be neglected)? [A rectangular array of compass needles aligns itself spontaneously, and this is sometimes used as a demonstration of "ferromagnetic" behavior on a large scale. It's a bit of a fraud, however, since the mechanism here is purely classical, and much weaker than the quantum mechanical **exchange forces** that are actually responsible for ferromagnetism.[13]]

More Problems on Chapter 6

! **Problem 6.22** In Prob. 6.4, you calculated the force on a dipole by "brute force." Here's a more elegant approach. First write $\mathbf{B}(\mathbf{r})$ as a Taylor expansion about the center of the loop:

$$\mathbf{B}(\mathbf{r}) \cong \mathbf{B}(\mathbf{r}_0) + [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla_0]\mathbf{B}(\mathbf{r}_0),$$

where $\mathbf{r}_0$ is the position of the dipole and ∇_0 denotes differentiation with respect to $\mathbf{r}_0$. Put this into the Lorentz force law (Eq. 5.16) to obtain

$$\mathbf{F} = I \oint d\mathbf{l} \times [(\mathbf{r} \cdot \nabla_0)\mathbf{B}(\mathbf{r}_0)].$$

Or, numbering the Cartesian coordinates from 1 to 3:

$$F_i = I \sum_{j,k,l=1}^{3} \epsilon_{ijk} \left\{ \oint r_l \, dl_j \right\} \left[\nabla_{0_l} B_k(\mathbf{r}_0) \right],$$

where ϵ_{ijk} is the **Levi-Civita symbol** ($+1$ if $ijk = 123$, 231, or 312; -1 if $ijk = 132$, 213, or 321; 0 otherwise), in terms of which the cross-product can be written $(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k$. Use Eq. 1.108 to evaluate the integral. Note that

$$\sum_{j=1}^{3} \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl},$$

where δ_{ij} is the Kronecker delta (Prob. 3.52).

[13]For an intriguing exception, see B. Parks, *Am. J. Phys.* **74**, 351 (2006), Section II.

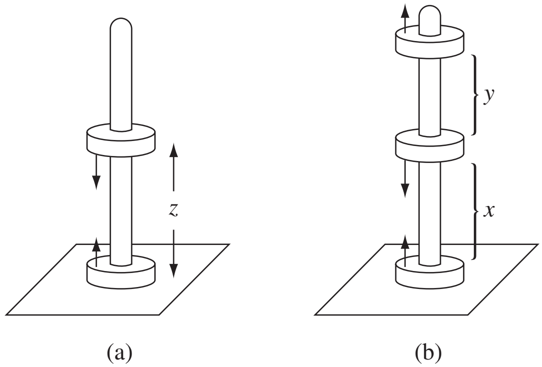


FIGURE 6.31

Problem 6.23 A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Fig. 6.31). Treat the magnets as dipoles, with mass m_d and dipole moment $\mathbf{m}$.

(a) If you put two back-to-back magnets on the rod, the upper one will "float"—the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?

(b) If you now add a *third* magnet (parallel to the bottom one), what is the *ratio* of the two heights? (Determine the actual number, to three significant digits.)
[*Answer:* (a) $[3\mu_0 m^2/2\pi m_d g]^{1/4}$; (b) 0.8501]

Problem 6.24 Imagine two *charged* magnetic dipoles (charge q, dipole moment $\mathbf{m}$), constrained to move on the z axis (same as Problem 6.23(a), but without gravity). Electrically they repel, but magnetically (if both $\mathbf{m}$'s point in the z direction) they attract.

(a) Find the equilibrium separation distance.

(b) What is the equilibrium separation for two *electrons* in this orientation.
[*Answer:* 4.72×10^{-13} m.]

(c) Does there exist, then, a stable bound state of two electrons?

Problem 6.25 Notice the following parallel:

$$\begin{cases} \nabla \cdot \mathbf{D} = 0, & \nabla \times \mathbf{E} = \mathbf{0}, \quad \epsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}, \quad \text{(no free charge)}; \\ \nabla \cdot \mathbf{B} = 0, & \nabla \times \mathbf{H} = \mathbf{0}, \quad \mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}, \quad \text{(no free current)}. \end{cases}$$

Thus, the transcription $\mathbf{D} \to \mathbf{B}$, $\mathbf{E} \to \mathbf{H}$, $\mathbf{P} \to \mu_0 \mathbf{M}$, $\epsilon_0 \to \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this, together with your knowledge of the electrostatic results, to rederive

(a) the magnetic field inside a uniformly magnetized sphere (Eq. 6.16);

(b) the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field (Prob. 6.18);

(c) the average magnetic field over a sphere, due to steady currents within the sphere (Eq. 5.93).

Problem 6.26 Compare Eqs. 2.15, 4.9, and 6.11. Notice that if ρ, $\mathbf{P}$, and $\mathbf{M}$ are *uniform*, the *same integral* is involved in all three:

$$\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Therefore, if you happen to know the electric field of a uniformly *charged* object, you can immediately write down the scalar potential of a uniformly *polarized* object, and the vector potential of a uniformly *magnetized* object, of the same shape. Use this observation to obtain V inside and outside a uniformly polarized sphere (Ex. 4.2), and $\mathbf{A}$ inside and outside a uniformly magnetized sphere (Ex. 6.1).

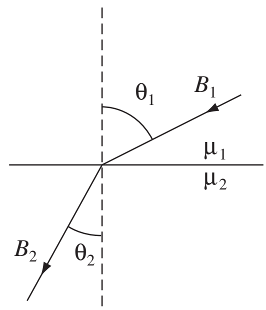


FIGURE 6.32

Problem 6.27 At the interface between one linear magnetic material and another, the magnetic field lines bend (Fig. 6.32). Show that $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$, assuming there is no free current at the boundary. Compare Eq. 4.68.

! **Problem 6.28** A magnetic dipole $\mathbf{m}$ is imbedded at the center of a sphere (radius R) of linear magnetic material (permeability μ). Show that the magnetic field inside the sphere ($0 < r \leq R$) is

$$\frac{\mu}{4\pi}\left\{\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2(\mu_0 - \mu)\mathbf{m}}{(2\mu_0 + \mu)R^3}\right\}.$$

What is the field *outside* the sphere?

Problem 6.29 You are asked to referee a grant application, which proposes to determine whether the magnetization of iron is due to "Ampère" dipoles (current loops) or "Gilbert" dipoles (separated magnetic monopoles). The experiment will involve a cylinder of iron (radius R and length $L = 10R$), uniformly magnetized along the direction of its axis. If the dipoles are Ampère-type, the magnetization is equivalent to a surface bound current $\mathbf{K}_b = M\,\hat{\boldsymbol{\phi}}$; if they are Gilbert-type, the magnetization is equivalent to surface monopole densities $\sigma_b = \pm M$ at the two ends. Unfortunately, these two configurations produce identical magnetic fields, at exterior points. However, the *interior* fields are radically different—in the first case $\mathbf{B}$ is in the *same* general direction as **M**, whereas in the second it is roughly *opposite* to **M**. The applicant proposes to measure this internal field by carving out a small cavity and finding the torque on a tiny compass needle placed inside.

Assuming that the obvious technical difficulties can be overcome, and that the question itself is worthy of study, would you advise funding this experiment? If so, what shape cavity would you recommend? If not, what is wrong with the proposal? [*Hint:* Refer to Probs. 4.11, 4.16, 6.9, and 6.13.]